Photonic Crystals

Molding the Flow of Light

Second Edition

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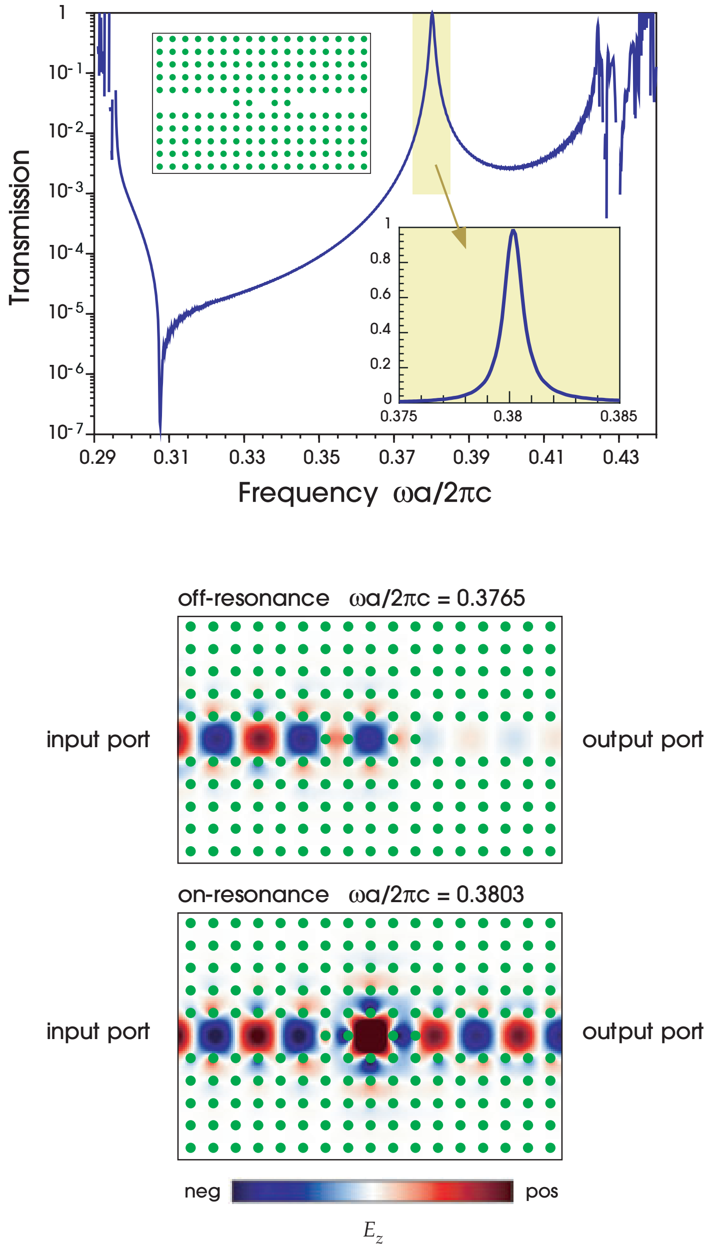

Figure 4: Waveguide–cavity–waveguide filter in rod crystal (inset, top). *Top:* Transmission spectrum, showing 100% peak at cavity resonance frequency ($\omega a/2\pi c = 0.3803$) with $Q = 410$; inset shows enlarged peak. Oscillations at low and high frequencies correspond to propagation outside the band gap, and sharp dip near $\omega a/2\pi c = 0.308$ corresponds to the zero-slope guided-band edge. *Bottom:* E_z field for transmission at a frequency 1% *below* resonance peak (upper), and exactly *at* resonance peak (lower).

This sharp peak means that the device acts as a *narrow-band filter*. The light is transmitted for frequencies near the resonant frequency of the cavity, and is reflected for somewhat lower or higher frequencies. The existence of the resonance peak conforms with intuition: near the resonant frequency, light from the input waveguide can couple into the cavity, and the cavity in turn can couple into the output waveguide. What may be surprising, however, is that the peak transmission is precisely 100%. The field pattern for transmission at resonance is shown in the bottom panel of figure 4. If we shift the frequency by only 1%, the transmission drops to less than 2%, corresponding to the fields in the middle panel. The fractional width $\Delta\omega/\omega_0$ at half-maximum (50% transmission) is precisely equal to $1/Q$, where Q is the quality factor of the cavity mode when excited internally. These and other properties of the transmission peak will be explained in the next section, Temporal Coupled-Mode Theory, in a more general setting.

Before we analyze the resonance peak, however, it is worth commenting on the other features of the transmission spectrum in figure 4. The sharp dip in the transmission at around $\omega a/2\pi c = 0.3075$ corresponds to the zero-slope band edge of the waveguide mode, where coupling light through the device is especially difficult. The oscillations at high and low frequencies correspond to frequencies outside the band gap, where energy propagates through the crystal instead of being confined to the waveguide and cavity. In a truly infinite system, light outside the gap would escape, but because we simulate this structure within a finite computational box, some light returns to the output waveguide, where interference results in an oscillating spectrum.

Temporal Coupled-Mode Theory

In order to analyze a broad range of devices, including the one in figure 4, we can exploit a powerful theoretical framework that falls into a general class of methods known as **coupled-mode theories**: one describes a system in terms of a set of idealized components (e.g. isolated waveguides and cavities) that are perturbed, or *coupled*, in some fashion. These methods are analogous to time-dependent perturbation theory in quantum mechanics, and they take many forms. Often, they are formulated as an expansion in the exactly computed eigenmodes of the idealized systems, providing a numerical result for a particular geometry.[8] The method we will discuss, **temporal coupled-mode theory**,[9] uses a more abstract formulation.

8 A classic presentation of this idea can be found in Marcuse (1991). Extensions to photonic crystals and further references can be found in Johnson et al. (2002*b*) and Povinelli et al. (2004).

9 See Haus (1984, ch. 7), or Suh et al. (2004) for a generalization. By *temporal coupled-mode theory*, we refer not merely to any time-dependent description, but to the more abstract formalism here, which was developed by Pierce (1954), Haus, and others as described in Haus and Huang (1991) and Louisell (1960). Related ideas also appear in the Breit–Wigner scattering theory of quantum mechanics (Landau and Lifshitz, 1977).

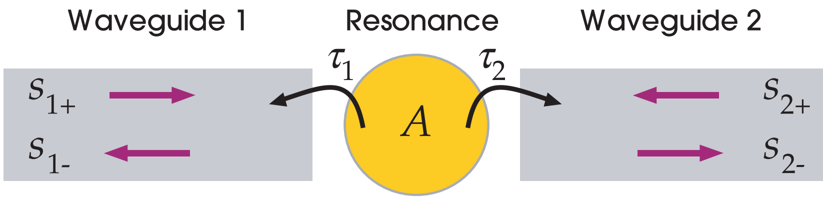

Figure 5: Abstract diagram showing the essential features of the filter from figure 4: a single-mode input waveguide 1, with input/output field amplitudes s_{1+}/s_{1-}; a single-mode output waveguide 2 with input/output field amplitudes s_{2+}/s_{2-}; and a single resonant mode of field amplitude A and frequency ω_0, coupled to waveguides 1 and 2 with lifetimes τ_1 and τ_2 ($\tau_1 = \tau_2$ in figure 4). The $s_{\ell\pm}$ are normalized so that $|s_{\ell\pm}|^2$ is power in the waveguide, and A is normalized so that $|A|^2$ is energy in the cavity.

In temporal coupled-mode theory, the system is considered as a set of essential components that are analyzed using only very general principles such as conservation of energy. Our building blocks will be *localized modes* (resonant cavities) and *propagating modes* (in waveguides). The result is a *universal* description of a certain class of devices. To obtain a quantitative result, the description is parameterized by a small number of unknowns such as the frequencies and decay rates of the resonant modes, which depend on the specific geometry and must be determined by a separate calculation.

This rather abstract idea is best understood by example. The structure of figure 4 is described in temporal coupled-mode theory as a resonant cavity connected to two single-mode waveguides (labelled 1 and 2), as depicted schematically in figure 5. There are no other places for the light to go; the rest of the crystal is ignored. The cavity mode has some resonant frequency ω_0 and decays with lifetimes τ_1 and τ_2 (defined more precisely below) into the two waveguides. In our structure, by symmetry, we must have $\tau_1 = \tau_2$, and it will turn out that this is the condition for 100% transmission on resonance. The key assumption of temporal coupled-mode theory (as with many other approximate methods) is that the coupling between the various elements is weak. In figure 5, for instance, we assume that the cavity energy leaks only slowly into the waveguides. We can imagine guaranteeing weak coupling by (for example) surrounding the cavity with a sufficient number of periods of the photonic crystal.

The temporal coupled-mode equations

We will now derive a set of equations describing the coupling of the cavity to the waveguides, in terms of the field amplitudes in those components. To do this, we will rely on five very general assumptions: weak coupling, linearity, time-invariance (i.e., the materials/geometry don't change over time), conservation of energy, and time-reversal invariance.[10] The most important of these

[10] Recall the section Time-Reversal Invariance of chapter 3.

is weak coupling; the latter four can actually be relaxed, as we will see in later sections.

We suppose that the fields in the cavity are proportional to some variable A. That is, since the eigenequation of the cavity completely determines the electric and magnetic fields up to some overall complex amplitude, we call this overall amplitude A (giving both magnitude and phase). Since the units of A can be chosen arbitrarily, we make the convenient choice that $|A|^2$ is the electromagnetic *energy* stored in the cavity.

We express the fields in the waveguide as the sum of incoming and outgoing waveguide modes, which are again defined up to an arbitrary complex amplitude $s_{\ell\pm}$ ($\ell = 1, 2$). Here, $s_{\ell+}$ is the amplitude of the mode in waveguide ℓ going *towards* the cavity, and $s_{\ell-}$ is the amplitude of the mode going *away from* the cavity. Again, since the units are arbitrary, we choose to make $|s_{\ell\pm}|^2$ the incoming (or outgoing) *power* in the waveguide modes.[11]

What are the equations governing these quantities? To begin with, consider the cavity mode by itself, with no incident power from the waveguides. Because the coupling is weak, it is safe to assume that the mode will decay exponentially over time with some lifetime τ. This is justified intuitively as follows.[12] If the mode hardly decays at all over one optical period, then the solution is approximately that of the lossless cavity. There is a fixed field pattern proportional to A, and the outgoing Poynting flux $\mathrm{Re}[\mathbf{E}^* \times \mathbf{H}]/2$ must therefore be proportional to $|A|^2$, the energy; since the rate of energy loss is proportional to the energy, an exponential decay ensues. Quantitatively, we require $\tau \gg 2\pi/\omega_0$, or $Q = \omega_0\tau/2 \gg \pi$. (In practice, we typically find temporal coupled-mode theory to be nearly exact for $Q > 30$, and often qualitatively accurate even for smaller Q.) If the cavity has two loss mechanisms, with decay constants τ_1 and τ_2, then the net lifetime is given by $1/\tau = 1/\tau_1 + 1/\tau_2$. The amplitude A satisfies a differential equation $dA/dt = -i\omega_0 A - A/\tau$, for which the solution is $A(t) = A(0)e^{-i\omega_0 t - t/\tau}$.

Now we include the waveguides. Input energy from $s_{\ell+}$ can couple into the cavity, or it can be reflected into $s_{\ell-}$ (or both). Energy from the cavity must also flow into $s_{\ell-}$. The most general linear, time-invariant equations relating these quantities, assuming weak coupling,[13] are

$$\frac{dA}{dt} = -i\omega_0 A - A/\tau_1 - A/\tau_2 + \alpha_1 s_{1+} + \alpha_2 s_{2+} \tag{2}$$

$$s_{\ell-} = \beta_\ell s_{\ell+} + \gamma_\ell A, \tag{3}$$

[11] More precisely, the time-dependent function $s_{\ell\pm}(t)$ is normalized so that its Fourier transform $\tilde{s}_{\ell\pm}(\omega)$ gives $|\tilde{s}_{\ell\pm}(\omega)|^2$ as the power per unit frequency at ω. This is because the waveguide field is not determined by a scalar amplitude at a single time: the field pattern is ω-dependent. However, we can largely ignore this subtlety because we are mostly interested in the response at either a single ω or for ω near the cavity resonance.

[12] A more formal argument can be based on the framework of leaky modes (Snyder and Love, 1983).

[13] Weak coupling is assumed here not only in that A is exponentially decaying, but also in that dA/dt depends simply on $s_{\ell+}$ multiplied by a constant. More generally, one could imagine a

for some proportionality constants α_ℓ, β_ℓ, and γ_ℓ. The constants α_ℓ and γ_ℓ represent the strength of the cavity–waveguide coupling, and β_ℓ is a reflection coefficient. It may seem that there are too many unknowns for this approach to be useful, but in fact we can eliminate *all* of the unknowns except for ω_0 and τ_ℓ.

The constants γ_1 and γ_2 can be determined using the conservation of energy. Consider the simplified case where $\tau_2 \to \infty$, so that the cavity is decoupled from waveguide 2, and suppose $s_{1+} = s_{2+} = 0$, so that there is no input energy. In this case, the cavity mode decays exponentially as $A(t) = A(0)e^{-i\omega_0 t - t/\tau_1}$, and thus the energy $|A|^2$ is decreasing. The only place for this energy to go is into the outgoing power $|s_{1-}|^2$. Thus, we must have

$$-\frac{d|A|^2}{dt} = \frac{2}{\tau_1}|A|^2 = |s_{1-}|^2 = |\gamma_1|^2|A|^2. \tag{4}$$

Therefore, $|\gamma_1|^2 = 2/\tau_1$, and since the phase of s_{1-} is arbitrary (it could represent the field amplitudes anywhere along the waveguide) we can choose $\gamma_1 = \sqrt{2/\tau_1}$. Similarly, if we let $\tau_1 \to \infty$, we find $\gamma_2 = \sqrt{2/\tau_2}$. But when both τ_1 and τ_2 are finite, does the decay into waveguide 2 affect γ_1 or vice versa? No, not if the decay rates are weak. The quantity γ_1 is already small; any change in γ_1 due to $1/\tau_2$ (another small quantity) is a second-order effect, which we will neglect.[14]

The constants α_ℓ and β_ℓ can be determined by time-reversal symmetry. We just saw that for $s_{\ell+} = 0$, the cavity mode decays and the output fields are given by $s_{\ell-} = \sqrt{2/\tau_\ell}A$. Time-reversal symmetry tells us we can obtain another valid solution of the equations by running the original solution backwards in time, and conjugating in order to retain an $e^{-i\omega_0 t}$ time dependence. That is, we must have a solution to equation (2) of the form $A(t) = A(0)e^{-i\omega_0 t + t/\tau}$ (exponentially growing) with input fields $s_{\ell+} = \sqrt{2/\tau_\ell}A$ and zero output fields $s_{\ell-} = 0$. Plugging this into equation (3), we immediately conclude that $\beta_\ell = -1$. (Thus, for $\tau_\ell \to \infty$ we get 100% reflection, $s_{\ell-} = -s_{\ell+}$, as we might expect. The minus sign is an artifact of our phase choice for γ_ℓ earlier and is not physically significant.) To determine α_1, we again employ the trick of taking $\tau_2 \to \infty$, in which case plugging $A(t)$ in equation (2) immediately gives $\alpha_1\sqrt{2/\tau_1}A = 2A/\tau_1$. Thus, $\alpha_\ell = \sqrt{2/\tau_\ell} = \gamma_\ell$, and we can again neglect higher-order effects of τ_2 on α_1 or vice versa, thanks to weak coupling.[15]

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convolution of $s_{\ell+}$ at different times, or equivalently a frequency-dependent $\alpha_\ell(\omega)$. Under weak coupling, however, only frequencies near ω_0 matter for coupling, in which case we can approximate $\alpha_\ell(\omega) \approx \alpha_\ell(\omega_0)$. Similarly for γ_ℓ.

[14] It is useful to see in this way that τ_1 and τ_2 are essentially independent quantities in the weak-coupling limit. Alternatively, conservation of energy implies that $2/\tau = |\gamma_1|^2 + |\gamma_2|^2$, and some authors simply *define* the individual decay rates by $2/\tau_\ell \triangleq |\gamma_\ell|^2$.

[15] This argument, while simple, is not strictly necessary. One can instead show that $\alpha_\ell = \gamma_\ell$ by exploiting an additional consequence of time-reversal symmetry and energy conservation, that the scattering matrix relating $s_{\ell+}$ and $s_{\ell-}$ is symmetric (Fan et al., 2003). We could also use reciprocity (Landau et al., 1984).

Finally, we have obtained the **temporal coupled-mode equations** for the system of figure 5:

$$\frac{dA}{dt} = -i\omega_0 A - \sum_{\ell=1}^{2} A/\tau_\ell + \sum_{\ell=1}^{2} \sqrt{\frac{2}{\tau_\ell}} s_{\ell+} \tag{5}$$

$$s_{\ell-} = -s_{\ell+} + \sqrt{\frac{2}{\tau_\ell}} A. \tag{6}$$

Note that we made no reference to the particular geometry of figure 5 in deriving these equations. They are valid for *any* filter satisfying our assumptions; the details matter only in determining the values of ω_0 and τ_ℓ. This approach is easily generalized to include more than two wave guides, radiative losses, and so on, as we will see. For now, however, we will leave the equations as they are and complete our analysis of the filter from figure 4.

The filter transmission

Given the coupled-mode equations (5) and (6), we can predict the transmission spectrum of any weakly-coupled waveguide-cavity-waveguide system. The transmission spectrum is simply the fractional output power $T(\omega) \triangleq |s_{2-}|^2/|s_{1+}|^2$ when $s_{2+} = 0$ (no input power from the right), as a function of the frequency ω.

Since frequency is conserved in a linear system, if the input oscillates at a fixed frequency ω, then the field everywhere must oscillate as $e^{-i\omega t}$, and $dA/dt = -i\omega A$. Plugging this, and $s_{2+} = 0$, into equations (5) and (6), we obtain:

$$-i\omega A = -i\omega_0 A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}} s_{1+} \tag{7}$$

$$s_{1-} = -s_{1+} + \sqrt{\frac{2}{\tau_1}} A \tag{8}$$

$$s_{2-} = \sqrt{\frac{2}{\tau_2}} A. \tag{9}$$

To solve for the transmission spectrum, divide equation (9) by s_{1+} and then solve for A/s_{1+} from equation (7). This gives:

$$T(\omega) = \frac{|s_{2-}|^2}{|s_{1+}|^2} = \frac{\frac{2}{\tau_2}|A|^2}{|s_{1+}|^2} = \frac{\frac{4}{\tau_1\tau_2}}{(\omega-\omega_0)^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)^2}. \tag{10}$$

This is the equation of a **Lorentzian peak** with a maximum at $\omega = \omega_0$. In the same way, we can derive the reflection spectrum:

$$R(\omega) = \frac{|s_{1-}|^2}{|s_{1+}|^2} = \frac{(\omega - \omega_0)^2 + \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)^2}. \tag{11}$$

It is easy to verify that $R(\omega) + T(\omega) = 1$ everywhere (energy is conserved), and that the reflection approaches 100% far from ω_0.

By inspecting equation (11) or equation (10), we see that $T(\omega_0) = 1$ *only* if $\tau_1 = \tau_2$, that is, when the cavity decays into the two waveguides at equal rates. In our photonic-crystal structure of figure 4, this equality is guaranteed by symmetry. The resonant reflection $R(\omega_0)$ is then zero. There are actually *two* sources of reflection—the direct reflection, and the light decaying backwards from the cavity—and at the resonance frequency these two reflections cancel exactly by destructive interference.

It is sometimes useful to write the transmission spectrum in terms of the quality factor Q instead of τ. The total lifetime is $1/\tau = 1/\tau_1 + 1/\tau_2 = 2/\tau_1$, and so $Q = \omega_0\tau/2$ implies $1/\tau_1 = 1/\tau_2 = \omega_0/4Q$. In this case, equation (10) becomes

$$T(\omega) = \frac{\frac{1}{4Q^2}}{\left(\frac{\omega - \omega_0}{\omega_0}\right)^2 + \frac{1}{4Q^2}}. \tag{12}$$

From equation (12), it follows that the fractional width $\Delta\omega/\omega_0$ at half-maximum ($T = 0.5$) is $1/Q$, as we observed in figure 4. In fact, if we were to plot equation (12) in figure 4, plugging in ω_0 and Q as determined by a small numerical computation (see appendix D), it would be nearly indistinguishable from the computed resonant peak shown in the inset.

To summarize, we have derived sufficient conditions for us to achieve a narrow-band filter with 100% transmission. We should have (i) a *symmetric* waveguide–cavity–waveguide system that is (ii) *single-mode* with (iii) *no other loss* mechanisms (such as radiation or absorption). Interestingly, the condition that our system be weakly coupled and amenable to the temporal coupled-mode theory is not really necessary, as we see in the next section. A photonic crystal provides the ideal situation for (iii), because it forbids all other radiative modes, whereas the losses that arise for an incomplete gap are analyzed in the section A Three-Dimensional Filter with Losses.

A Waveguide Bend

The applicability of temporal coupled-mode theory and figure 5 to the photonic-crystal filter of figure 4 is clear. Similar ideas can help us to understand