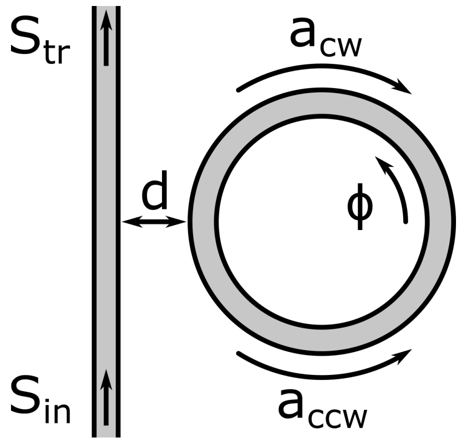

Figure 1: Schematic diagram of the situation discussed in the assignment.

Nanophotonics 2018 - Problem Set Microcavities

Class date:	Tuesday, 8th May 2018
Due date:	Thursday, 24th May 2018
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An optical fibre is positioned at a distance d from a microtoroid (ring) in vacuum. Light at the resonance frequency of a whispering gallery mode (WGM) in the toroid is sent into the waveguide, as in figure 1. At $d = d_{\mathrm{crit}}$ critical coupling is reached between the fibre and the ring resonator.

The transmitted light amplitude s_{tr} through the fibre is related to the input light amplitude s_{in} and the intracavity light amplitude a_{cw} as $s_{\mathrm{tr}} = s_{\mathrm{in}} - \sqrt{\kappa_{\mathrm{ex}}} a_{\mathrm{cw}}$. The light propagating forward through the fibre only couples to the clockwise-propagating WGM. This WGM corresponds to a travelling wave of the form $\tilde{a}_{\mathrm{cw}} \sim \exp(\mathrm{i}m\phi - \mathrm{i}\omega_{\mathrm{c}}t)$ with azimuthal coordinate ϕ and integer m. The clockwise-propagating mode in turn only couples to forward propagating light in the fibre, so there is no reflection in the fibre.

1. Give an expression for the transmission $T = |s_{\mathrm{tr}}|^2/|s_{\mathrm{in}}|^2$ as a function of frequency detuning Δ, coupling rate κ_{ex}, and damping rate κ. How much light is transmitted on resonance at critical coupling? What happens to the rest of the light? What is the transmission at critical coupling for detuning $\Delta = \kappa/2$?

If a nanoparticle (10–100 nm, e.g. a small colloid or a virus particle) attaches to the toroid surface, it can disturb the electric field distribution in the toroid and scatter some of the light. The particle can be detected by looking at the frequency shift of a resonance when the particle binds to the toroid. However, also temperature fluctuations can cause the resonance to shift, so this method suffers from thermal noise and other environmental disturbances.

Another way to detect the particle is to look at *mode splitting*: because the particle scatters the light, it can scatter light from the clockwise-propagating WGM into the counterclockwise-propagating WGM. To account for a scattering with strength γ, we can add another term in the equations of motion for the intra-cavity amplitudes:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_{\mathrm{cw}}(t) = (\mathrm{i}\Delta - \kappa/2)a_{\mathrm{cw}}(t) + \mathrm{i}\frac{\gamma}{2}a_{\mathrm{ccw}}(t) + \sqrt{\kappa_{\mathrm{ex}}}s_{\mathrm{in}}(t) \quad (1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a_{\mathrm{ccw}}(t) = (\mathrm{i}\Delta - \kappa/2)a_{\mathrm{ccw}}(t) + \mathrm{i}\frac{\gamma}{2}a_{\mathrm{cw}}(t). \quad (2)$$

2. Show that the above system of coupled equations of motion can be rewritten as a set of *uncoupled* equations by considering the normal modes $a_\pm = \frac{1}{\sqrt{2}}(a_{\mathrm{cw}} \pm a_{\mathrm{ccw}})$ instead. What are the resonance frequencies $\omega_\pm$ and the damping rates $\kappa_\pm$ of the normal modes $a_\pm$?

3. Work out the transmission at the resonance frequencies of $a_\pm$ at critical coupling. In the limit $\gamma \ll \kappa$ this system reduces to the one in question 1. What is the transmission at the new resonance frequencies in the limit $\gamma \gg \kappa$?

The difference between these two frequencies does not depend on the temperature of the toroid. This means that nanoparticles can be more easily detected in experiments with environmental noise by monitoring the splitting.

The frequency splitting $\gamma/2\pi$ depends on the particle polarisability α_{SI}[1] and the position of the particle in the optical mode profile, i.e. on where on the toroid it adsorbs. It can be approximated by

$$\gamma = \frac{\alpha_{\mathrm{SI}} f^2(\mathbf{r})\omega_{\mathrm{c}}}{\epsilon_0 V}, \quad (3)$$

where $f^2(\mathbf{r})$ is a dimensionless quantity that follows the spatial intensity profile of a mode, normalised such that $\int f^2(\mathbf{r})d\mathbf{r} = V$. The particle adsorbs at position $\mathbf{r}$ and V is the mode volume.

We want to detect a KCl particle that is small enough that we can assume it to cause Rayleigh scattering described by a static polarisability.

4. Assume a whispering gallery mode with a mode volume of $1.5 \times 10^{-16}\,\mathrm{m}^3$ driven resonantly at a wavelength of 670 nm. A particle adheres at a position such that $f^2(\mathbf{r}) = 1$. Estimate the smallest KCl particle diameter for which the mode splitting can be resolved if the quality factor of the toroid is 2×10^8.

Figure 2 shows a measurement where mode splitting is observed when adding more and more KCl particles. The graph shows behaviour that can not be seen in the above equations: the resonance not only splits but the two modes also become broader. Moreover, there is a difference between the two normal modes, i.e. between the even and odd superposition of the clockwise and counterclockwise modes.

5. Write the spatial profile of the normal modes as a function of azimuthal coordinate and time. What effect did we neglect? How can it be that the adsorption of a particle affects one normal mode more strongly than the other?

[1] We describe polarisability in SI units, as on slide 12 of class 6.

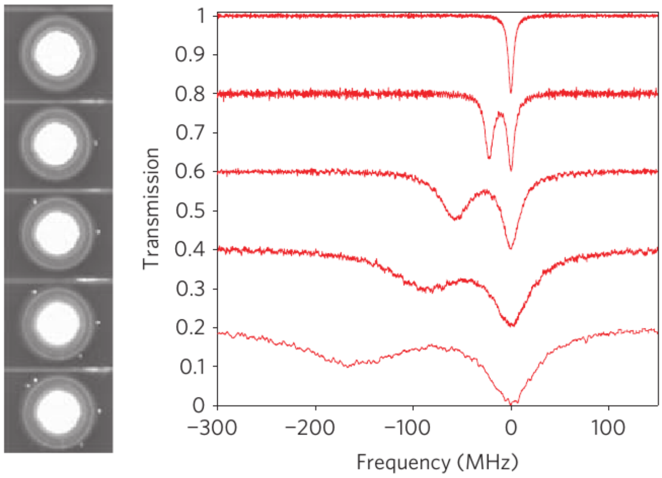

Figure 2: Series of normalised transmission spectra and the corresponding optical images (assisted by a visible light laser) recorded without nanoparticles (top trace) and with four successive depositions of KCl nanoparticles. The spectra are vertically shifted for clarity.

The broadening represents the loss rate of energy stored in the mode caused by the presence of a scatterer, where scattered power is the product of an effective scattering cross-section and the energy flow through that area.

6. Assuming Rayleigh scattering, show that the broadening of the resonance can be estimated by
$$\kappa_{\text{extra}} = \frac{\omega_c k^3 \alpha_{\text{SI}}^2 f^2(\mathbf{r})}{6\pi\epsilon_0^2 V}. \tag{4}$$

7. Explain that measuring *both* the broadening and the splitting due to the adsorption of a single particle provides a good way to estimate the *absolute* diameter of the particle.