# Lecture Nanophotonics
# Coupled light-matter systems Assignment

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| *Class dates:* | *Tuesday 15 May 2018* |
| *Due date:* | *Thursday 24 May 2018* |
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## 1 Coupled harmonic oscillators

Consider the equations of motion of two coupled damped oscillators 1 and 2:

$$\begin{aligned}\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+\gamma_1\frac{\mathrm{d}x_1}{\mathrm{d}t}+\omega_1^2x_1-\Omega^2x_2=0\\ \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2}+\gamma_2\frac{\mathrm{d}x_2}{\mathrm{d}t}+\omega_2^2x_2-\Omega^2x_1=0\end{aligned} \tag{1}$$

Under certain approximations and assuming solutions of the form: $x_{1,2}=C_{1,2}\,e^{-i\omega t}$, the above system can be mapped into an eigenvalue problem $|H-\omega I|=0$ with a non-Hermitian Hamiltonian:

$$H=\begin{pmatrix}\omega_1-i\frac{\gamma_1}{2} & -\frac{\kappa}{2}\\ -\frac{\kappa}{2} & \omega_2-i\frac{\gamma_2}{2}\end{pmatrix} \tag{2}$$

(a) Show that the eigenvalues of this Hamiltonian, which correspond to the eigenfrequencies of the coupled oscillators, can be expressed as $\omega_\pm=\bar{\omega}\pm\frac{1}{2}\sqrt{\Delta^2+\kappa^2}$, where $\bar{\omega}=\frac{\omega_1+\omega_2}{2}-i\frac{\gamma_1+\gamma_2}{4}$, $\Delta=\omega_1-\omega_2-i\frac{\gamma_1-\gamma_2}{2}$, $\kappa=\frac{\Omega^2}{\bar{\omega}}$ (assume $\omega_1\approx\omega_2\approx\omega$).

(b) Assuming $\omega_1=1$, $\gamma_1=0.01$, $\gamma_2=0.02$, plot $\mathrm{Re}\{\omega_\pm\}$ and $\mathrm{Im}\{\omega_\pm\}$ as a function of $\omega_2$ in the range $\omega_2\in[0.995,1.005]$ at three $\kappa$: $\kappa=0.004,0.005,0.006$ (plot $\mathrm{Re}\{\omega_\pm\}$ in one figure, $\mathrm{Im}\{\omega_\pm\}$ in another figure).

(c) Assuming $\omega_1=\omega_2=1$, $\gamma_1=0.01$, $\gamma_2=0.02$, plot $\mathrm{Re}\{\omega_\pm\}$ and $\mathrm{Im}\{\omega_\pm\}$ as a function of $\kappa$ in the range $\kappa\in[0,0.01]$ (plot $\mathrm{Re}\{\omega_\pm\}$ in one figure, $\mathrm{Im}\{\omega_\pm\}$ in another figure). Find the $\kappa_c$ at which $\mathrm{Re}\{\omega_+\}=\mathrm{Re}\{\omega_-\}$ and $\mathrm{Im}\{\omega_+\}=\mathrm{Im}\{\omega_-\}$.

In non-Hermitian quantum mechanics, the condition where $\mathrm{Re}\{\omega_+\}=\mathrm{Re}\{\omega_-\}$ and $\mathrm{Im}\{\omega_+\}=\mathrm{Im}\{\omega_-\}$ is known as the **exceptional point**.
Now consider a driving force $F\,e^{-i\omega t}$ acting on oscillator 1. The equations of motion become:

$$\begin{aligned}\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+\gamma_1\frac{\mathrm{d}x_1}{\mathrm{d}t}+\omega_1^2x_1-\Omega^2x_2=Fe^{-i\omega t}\\ \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2}+\gamma_2\frac{\mathrm{d}x_2}{\mathrm{d}t}+\omega_2^2x_2-\Omega^2x_1=0\end{aligned} \tag{3}$$

(d) Assuming $x_{1,2}=C_{1,2}e^{-i\omega t}$, prove the expressions for the complex amplitudes $C_{1,2}$ are:

$$\begin{aligned}C_1=\frac{(\omega_2^2-\omega^2-i\omega\gamma_2)F}{(\omega_2^2-\omega^2-i\omega\gamma_2)(\omega_1^2-\omega^2-i\omega\gamma_1)-\Omega^4}\\ C_2=\frac{\Omega^2F}{(\omega_2^2-\omega^2-i\omega\gamma_2)(\omega_1^2-\omega^2-i\omega\gamma_1)-\Omega^4}\end{aligned} \tag{4}$$

(e) The complex amplitude $C_{1,2}$ can be expressed in terms of its modulus and phase as $C_{1,2}=|C_{1,2}|\,e^{i\phi_{1,2}}$. Assuming $\omega_1=1$, $\omega_2=1.05$, $\gamma_1=0.01$, $\gamma_2=0.001$ and $\Omega=0.2$, make the following plots in the range $\omega\in[0.9,1.1]$:

(i) $|C_1|$ and $|C_2|$;

(ii) $\phi_1$, $\phi_2$ and $\phi_1 - \phi_1$;

Explain what happens to $|C_1|$ at $\omega_2$, based on the behavior of the phases of the coupled oscillators. (Hint: think of what interferes with what.)

## 2 Single non-linear oscillator

The dynamics of the intra-cavity field of a Kerr non-linear microcavity under continuous-wave driving can be described by the following equation:

$$i\dot{\alpha} = (\omega_0 - i\frac{\gamma}{2})\alpha + U|\alpha|^2\alpha + Fe^{-i\omega_L t} \tag{5}$$

Here, $\omega_0$ is the cavity resonance frequency; $\gamma$ is loss rate; $U$ is the interaction energy; $F$ and $\omega_L$ are the driving amplitude and frequency. To get rid of the time harmonics of the driving force, we move to a frame rotating at the driving frequency $\omega_L$, such that Eq.(5) becomes:

$$i\dot{\alpha} = (-\Delta - i\frac{\gamma}{2})\alpha + U|\alpha|^2\alpha + F \tag{6}$$

where $\Delta = \omega_L - \omega_0$ is the laser-cavity detuning.
Now let's focus on the steady state solution(s), given by setting $\dot{\alpha} = 0$.

(a) Show that the steady-state solution satisfies the following equation:

$$|F|^2 = U^2n^3 - 2\Delta Un^2 + (\Delta^2 + \frac{\gamma^2}{4})n \tag{7}$$

where $n = |\alpha|^2$ is the steady-state density.

(b) Assuming $\gamma = 0.01$, $U = 0.0075\gamma$, $\omega_0 = 1$, use Eq.(7) to make the plot of $n$–$|F|^2$ in the range of $|F|^2 \in [23\gamma, 30\gamma]$ at two different $\Delta$: (i)$\Delta = 0.85\gamma$, (ii)$\Delta = 0.9\gamma$ (plot two curves in the same figure).

(c) As shown in the plots of (b), there exist three steady states at certain $\Delta$ and $F$ (only two steady states are stable, the so called **bistability**). Also, there exist a critical detuning $\Delta_c$ for the onset of the bistability. Derive the analytical formula for $\Delta_c$.

(Hint: think about $\partial|F|^2/\partial n$)

(d) Evaluate the stability of the steady state solutions in the subquestion (b)(ii). The procedure is as follows:

(i) Introduce a linear fluctuation around the steady state, i.e. $\tilde{\alpha} = \alpha + \delta\alpha$, and plug it in Eq.(6). Then, expand the equation and only retain terms which are linear in the fluctuation $\delta\alpha$. Show that the differential equation for $\delta\alpha$ can be written into matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\alpha \\ \delta\alpha^*\end{pmatrix} = \mathbf{A}\begin{pmatrix}\delta\alpha \\ \delta\alpha^*\end{pmatrix} \tag{8}$$

$$\mathbf{A} = \begin{pmatrix}(\Delta - 2U|\alpha|^2)i - \frac{\gamma}{2} & -iU(\alpha)^2 \\ iU(\alpha^*)^2 & (2U|\alpha|^2 - \Delta)i - \frac{\gamma}{2}\end{pmatrix} \tag{9}$$

(ii) The Hurwitz criterion states that if $\mathrm{Tr}\{\mathbf{A}\} < 0$ and $\det\{\mathbf{A}\} > 0$ (where $\mathrm{Tr}\{\mathbf{A}\}$ and $\det\{\mathbf{A}\}$ are the trace and the determinant of the matrix $\mathbf{A}$) the fluctuation approaches zero at $t \to \infty$, meaning that the steady state is stable. Based on this, numerically evaluate all the steady state solutions in subquestion (b)(ii) and remake the plot, in which the un-stable solutions should be plotted differently (e.g. a different color or line style).