Lecture Nanophotonics
Plasmonics and Metamaterials Assignment

Class dates:	*Thursday 10 April 2018*
Due date:	*Thursday 17 April 2018*
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Note:

- We will only work with SI units.
- We use the convention in which the time-dependent part of a time-harmonic variable $f(\omega)$ is given as $e^{-i\omega t}$, with ω the angular frequency. For instance, $E(t) = Ee^{-i\omega t}$
- The notations $\Im(a)$ and $\Re(a)$ are used for respectively the real and imaginary part of a complex number a.
- It may help to plot graphs on a log scale (y-axis).

1 Kramers-Kronig relations

The Kramers-Kronig equations constitute an important relation between the real and imaginary part of any linear response function in a physical system. They can be derived using only causality, and are therefore of very general validity.
Assume a system in which a quantity P (e.g. macroscopic polarization) is related to an external quantity E (e.g. electric field) by a linear response function χ in the frequency domain:

$$P(\omega) = \chi(\omega)\, E(\omega) \tag{1}$$

Then, in the time domain:

$$P(t) = \int_{-\infty}^{-\infty} \chi(t - t')\, E(t')dt' \tag{2}$$

Because this is a physical system, in the time domain it must hold that:

$$\{P(t), \chi(t), E(t)\} \in \mathbb{R} \qquad \text{for all } t. \tag{3}$$

Furthermore, causality implies that

$$\chi(t) = 0 \qquad \text{for} \qquad t \leq 0 \tag{4}$$

because there can not be a response before there is a cause.

a) Write $\chi(\omega)$ in terms of $\chi(t)$ and split in its real and imaginary part $\chi'(\omega)$ and $\chi''(\omega)$.

b) Now we will use a mathematical trick to simplify the previous result. As with any function, $\chi(t)$ can be expressed in its even and odd parts $\chi_e(t)$ and $\chi_o(t)$ as

$$\chi(t) = \chi_e(t) + \chi_o(t) \tag{5}$$

Use causality to express $\chi_e(t)$ in terms of $\chi_o(t)$.

Tip: if you find this difficult, imagine $\chi(t)$ is just a unit step function, write down $\chi_e(t)$ and $\chi_o(t)$ and see what the relation between the two is.

c) Use Eq. (5), the results from questions 1 a) and 1 b) and the convolution theorem to show that

$$\chi'(\omega) = [i\,\chi''(\omega)] \star \left[\int_{-\infty}^{\infty} \mathrm{sgn}(t)e^{i\omega t}dt\right] \tag{6}$$

where $\star$ denotes a convolution.

d) Show that

$$\chi'(\omega) = \int_{-\infty}^{\infty} \frac{2\chi''(\omega')}{\omega' - \omega} d\omega' \tag{7}$$

Hint: prove the relation:

$$\lim_{\epsilon \downarrow 0} \int_0^{\infty} e^{(i\omega - \epsilon)t} = \lim_{\epsilon \downarrow 0} \frac{1}{i\omega - \epsilon} e^{(i\omega-\epsilon)t}\Big|_{t=0}^{\infty} = \frac{i}{\omega} \tag{8}$$

e) You have derived the first of the two Kramers-Kronig relations. Is it possible to have a narrow frequency window of high absorption (e.g. $\chi''(\omega) = C_1\delta(\omega - \omega_0)$ with $C_1 \in \mathbb{R}$) and no dispersion (i.e. $\chi'(\omega)$ is constant for all ω)?

f) The second Kramers-Kronig relation can be derived similarly and is (you do not need to derive it):

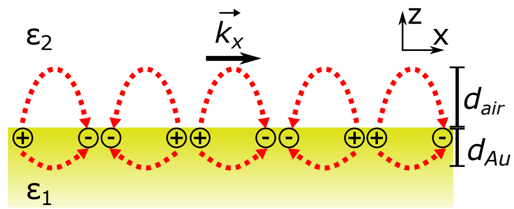
$$\chi''(\omega) = -\int_{-\infty}^{\infty} \frac{2\chi'(\omega')}{\omega' - \omega} d\omega' \tag{9}$$

Is it possible to have a window of non-zero $\chi'(\omega)$ (e.g. $\chi'(\omega) = C_1(\Theta(\omega - \omega_1) - \Theta(\omega - \omega_2))$ with $C_1 \in \mathbb{R}$, $\omega_1 < \omega_2$ and $\Theta(x)$ the Heaviside step function) and no absorption?

2 Surface plasmon dispersion relation

The dispersion relation of a Surface Plasmon Polariton (SPP) at a metal-insulator interface is given by:

$$k_x^2 = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{\omega^2}{c^2} \tag{10}$$



a) Plot the dispersion relation (real and imaginary part of k_x) of a SPP at a gold-air interface as a function of ω (0 to 1.5 $\omega_{p,\mathrm{Au}}$). For the permittivity of gold, use a Drude model with the following parameters: plasma frequency $\omega_{p,\mathrm{Au}} = 2\pi \cdot 2165$ THz, damping frequency $\gamma_{\mathrm{Au}} = 2\pi \cdot 15.9$ THz.

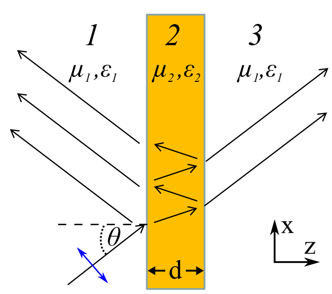
b) Plot the phase velocity and the group velocity of the SPP as a function of ω and compare both to the speed of light. What happens to the velocities when $\Re(k_x(\omega))$ reaches its maximum, and at $\omega = \omega_{p,\mathrm{Au}}$?

c) The propagation length l is related to the wavevector k_x in the following way: $l = \frac{1}{\Im(2 \cdot k_x)}$. Plot l as a function of ω and comment on special features and their origin.

d) How deep does the electric field reach into the air medium and how deep into the gold medium? Plot d_{air} and d_{Au} as a function of ω and explain.

3 Negative refraction makes a perfect lens

In this exercise, we take a look at materials with simultaneously negative permittivity ϵ and permeability μ. These so-called *metamaterials* (MMs) gained large attention due to their unusual electromagnetic properties, such as negative index of refraction and perfect imaging. Here, we want to consider the case of a slab of a MM (such as proposed by Veselago and Pendry) made of $\epsilon = -1$ and $\mu = -1$ material with a thickness d and understand what makes it a *perfect lens*. One of the most provoking claims of Veselago and Pendry is that waves experience a negative phase delay inside of such media and evanescent waves get exponentially amplified. As we will see, some of these intriguing optical properties are already evident from the transmission properties.



a) Derive the formula of the total amplitude transmission t of a plane wave incident from the left side on a slab with optical properties ϵ_2, μ_2 and thickness d, surrounded by a material with ϵ_1 and μ_1 and express it in terms of the transmission and reflection coefficients of the two interfaces of the slab.

Hint 1: Consider all light paths contributing to the total transmission, including multiple reflected waves inside of the slab.
Hint 2: Use the infinite geometric series formula:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}. \tag{11}$$

b) Consider a dielectric slab (a so-called 'etalon') with $d = 400\,\text{nm}$, $\epsilon_2 = 10$ and $\mu_2 = 1$ surrounded by air. Apply the derived formula to plot the intensity transmission $|t|^2$ through this slab as a function of frequency ω in the visible regime. Assume p-polarized light, which is incident at $\theta = 0°$ in the x-z plane ($k_{y,1} = 0$). What do you see? What is the physical origin of the fluctuations?
Hint 1: The correct solution of a) is:

$$t = \frac{t_{23} t_{12} e^{id \cdot k_{z,2}}}{1 - r_{21} r_{23} e^{2i \cdot d \cdot k_{z,2}}}. \tag{12}$$

Hint 2: The Fresnel transmission and reflection coefficients for the interface between material 1 and 2 for p-polarized light are:

$$t_{12}^p = \frac{2\epsilon_2 k_{z,1}}{\epsilon_2 k_{z,1} + \epsilon_1 k_{z,2}} \sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \epsilon_2}}, \quad r_{21}^p = \frac{\epsilon_2 k_{z,1} - \epsilon_1 k_{z,2}}{\epsilon_2 k_{z,1} + \epsilon_1 k_{z,2}}, \tag{13}$$

where $k_{z,i} = \sqrt{k_i^2 - k_{||}^2}$ and $k_i = \frac{\omega}{c}\sqrt{\epsilon_i \mu_i}$ with material indeces $i = 1, 2$.

c) Use the same equation as in b) to plot $|t|^2$ and $\arg(t)$ as a function of the normalized parallel wavevector in the interval $\{0 \leq k_{||}/k_1 \leq 2\}$ at $\omega = 2\pi \cdot 375\,\text{THz}$. Assume that all materials are air ($\epsilon_1 = \epsilon_2 = 1$ and $\mu_1 = \mu_2 = 1$) and $d = 50\,\text{nm}$. What do you see? What does $k_{||} > 1$ mean for the incidence angle and k_z? What is the physical significance of $\arg(t)$?

d) Describe qualitatively how the amplitude of propagating ($\omega^2/c^2 > k_{||}^2$) and evanescent ($\omega^2/c^2 < k_{||}^2$) waves changes as a function of distance if the slab is made of air (*hint: use equation 12*). And what happens in the case of negative index MM ($\epsilon_2 = -1$, $\mu_2 = -1$)? What are the directions of $k_{z,2}$, Poynting vector, phase advancement and energy flux in the negative index MM?

e) Make the same plots as in c) at $\omega = 2\pi \cdot 375\,\text{THz}$ for a slab of MM with $\epsilon_2 = -1$ and $\mu_2 = -1$ and $d = 50\,\text{nm}$. Pay attention to the orientation of $k_{z,2}$. What is the surprising part about these plots as compared to c)?