Photonic Crystals 2018

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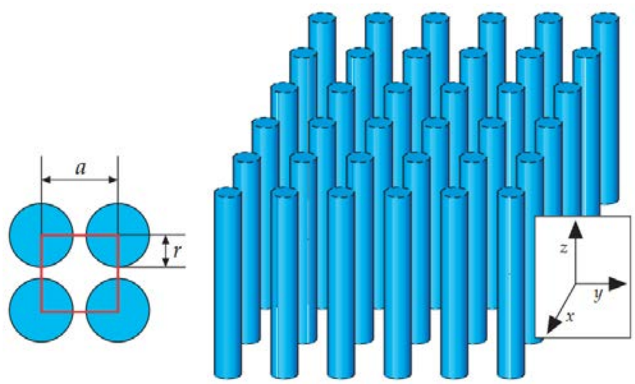
1. Consider a system as shown in the figure below. This problem illustrates how symmetry in the system may be used to infer the symmetry of the modes supported by the system. In this problem we focus on mirror symmetry. Use the fact that the electric field $\vec{E}$ transforms like a vector and the magnetic field $\vec{H}$ transforms like a pseudo-vector under mirror transformation. This means for $\vec{E}$, only the components perpendicular to the mirror plane change signs. On the other hand for $\vec{H}$, only the components parallel to the mirror plane change signs. Now using mirror symmetry about the plane perpendicular to the z-axis, find out which components of $\vec{E}$, $\vec{H}$ exist for odd modes and even modes with respect to this plane. Remember that odd modes obey $\vec{F}(\boldsymbol{r}) = -\vec{F}(-\boldsymbol{r})$ and even modes obey $\vec{F}(\boldsymbol{r}) = \vec{F}(-\boldsymbol{r})$, where $\vec{F}$ stands for either $\vec{E}$ or $\vec{H}$.



2. Consider a periodic system: : $\epsilon(\boldsymbol{r}) = \epsilon(\boldsymbol{r}+\boldsymbol{R})$, where for a square lattice $\boldsymbol{R}$ is any linear combination of the lattice vectors $\boldsymbol{R} = na\hat{\boldsymbol{x}} + ma\hat{\boldsymbol{y}}$,with $\boldsymbol{r}$ an arbitrary vector in the x,y-plane and $\boldsymbol{x}$ and $\boldsymbol{y}$ unit vectors and $n, m = 0, \pm1, \pm2$ ...
The Bloch theorem states that modes in a periodic system with reciprocal lattice vectors **G** ($e^{i\boldsymbol{G}.\boldsymbol{R}} = 1$), can be written as $\boldsymbol{E}_K(\boldsymbol{r}) = \boldsymbol{u}(\boldsymbol{r})e^{i\boldsymbol{K}.\boldsymbol{r}}$ where $\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{R}) = \boldsymbol{u}(\boldsymbol{r})$. Using the periodicity of $\boldsymbol{u}(\boldsymbol{r})$, show that $\boldsymbol{E}_{\boldsymbol{K}}(\boldsymbol{r}) = \sum_{\boldsymbol{G}} \boldsymbol{u}_{\boldsymbol{K},\boldsymbol{G}} e^{i(\boldsymbol{K}+\boldsymbol{G}).\boldsymbol{r}}$.

3. Now consider a 1d problem consisting of system periodic in the y-direction and homogenous in the x and z directions. Use the wave equation $\nabla \times \nabla \times \boldsymbol{E} - \frac{\omega^2}{c^2}\mu\epsilon\boldsymbol{E} = 0$ and assuming a mode propagating in the y-direction with electric field in the z-direction, and $\boldsymbol{G} = \frac{2\pi}{a}\hat{\boldsymbol{y}}$, use the result for the electric field obtained in problem 2 to show that

$$\sum_{\boldsymbol{G}} u_{Ky,G}\left(K_y+G\right)^2 e^{i(K_y+G)y} - \frac{\omega^2}{c^2}\mu\epsilon \sum_{\boldsymbol{G}} u_{Ky,G} e^{i(K_y+G)y} = 0$$

4. Now taking $\mu = 1$ and using the periodicity of the dielectric constant $\epsilon(y) = \epsilon(y + a)$, use the Fourier expansion of ϵ to show that

$$\sum_{\boldsymbol{G}} u_{Ky,G} \left(K_y + G\right)^2 e^{(i(K_y+G)y)} - \frac{\omega^2}{c^2}\mu \sum_{\boldsymbol{G'}} \epsilon_{\boldsymbol{G'}} e^{iG'y} \sum_{\boldsymbol{G}} u_{Ky,G} e^{(i(K_y+G)y)} = 0$$

where ϵ_G are Fourier coefficients.

5. Using the above result show that

$$u_{Ky,G}\left(K_y + G\right)^2 - \left(\frac{\omega}{c}\right)^2 \sum\nolimits_{G'} u_{Ky,G-G'}\epsilon_{G'} = 0$$

6. Assuming that only Fourier coefficients $\epsilon_0, \epsilon_{-1}, \epsilon_1$ are significant, write the above equation for $G = 0, \frac{2\pi}{a}, \frac{-2\pi}{a}$. Using these 3 equations and choosing $K_y \approx \frac{\pi}{a}, \left|K_y - G\right| = K_y$ (Bragg condition) show that $u_{K,0}, u_{K,-G}$ dominantly couple to each other.

7. Using the above result write a set of 2 coupled equations for $u_{K,0}, u_{K,-G}$. Using these equations, obtain the dispersion relation (equation connecting ω, k)

$$\left(\frac{\omega}{c}\right)^4 (\epsilon_0^2 - \epsilon_1\epsilon_{-1}) - \left(\frac{\omega}{c}\right)^2 \epsilon_0 \left(\left(K_y - G\right)^2 + K_y{}^2\right) + K_y{}^2\left(K_y - G\right)^2 = 0$$

8. Show that when the Bragg condition is exactly satisfied ($K_y = \pm\frac{\pi}{a}$) , two roots for ω^2 may be obtained:

$$\omega_+^2 = \frac{\pi^2 c^2}{a^2(\epsilon_0 - |\epsilon_1|)} \quad \text{and} \quad \omega_-^2 = \frac{\pi^2 c^2}{a^2(\epsilon_0 + |\epsilon_1|)}$$

9. The 1d stack was analyzed with an alternative, exact method, the transfer matrix method in class and the following dispersion was obtained :

$$Kd = arccos\frac{1}{2}[2\cos(k_1 d_1)\, cos(k_2 d_2) - \left(\frac{k_2}{k_1} + \frac{k_1}{k_2}\right) sin(k_1 d_1)\sin(k_2 d_2)$$

$d_1 + d_2 = a$. Consider a quarter wave stack such that $n_1 d_1 = n_2 d_2$. Plot this dispersion varying $\frac{\omega a}{2\pi c}$ from 0 to 2 and compare with dispersion obtained in problem 7 where now $K_y = K$. Show the regions where you expect the bandgap. For the Fourier coefficients of the quarter wave stack use $\epsilon_0 = n_1 n_2$ and $\frac{|\epsilon_1|}{\epsilon_0} = \frac{4}{\pi}\frac{|n_2 - n_1|}{n_2 + n_1}$. You can use $n_1 = 1, n_2 = 1.5$.