Lecture Nanophotonics - Nanoscale antennas Assignment

Class Dates:	*April 24*
Due:	*May 1*
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Note:

- Be aware that this is one of the largest assignments in the course. It would be wise not to postpone until a few days before the deadline.
- Throughout the whole assignment we only deal with non magnetic dielectric media, so we always assume $\mu = 1$.
- We will only work with SI units.
- Possible useful expressions are:
  - We use the convention in which the time-dependent part of a time-harmonic variable $f(\omega)$ is given as $e^{-i\omega t}$, with ω the angular frequency.
  - Electric field at position $\mathbf{r}$ created by a dipolar moment $\mathbf{p}$ located at $\mathbf{r}'$is:
  $$\mathbf{E}(\mathbf{r}) = \omega^2 \mu\mu_0 \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{p}\,, \tag{1}$$
  where $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)$ is the free space dyadic Green's function.
  - The extinction and scattering cross sections of a dipolar scatterer with (scalar) polarizability α are:
  $$\sigma_{ext} = \frac{k}{\varepsilon_0}\Im(\alpha), \tag{2}$$
  $$\sigma_{scat} = \frac{k^4}{6\pi\varepsilon_0^2}|\alpha|^2, \tag{3}$$
  where $k = (2\pi/\lambda)\sqrt{\varepsilon\mu}$ is the magnitude of the wave vector, λ is the wavelength and α relates the scalar dipolar moment p to the scalar electric field E_0 at the dipole position as $p = \alpha\, E_0$.
- For a brief explanation of what a Lorentzian is, see the document 'Lorentzian and the good cavity approximation'.
- As a final notation remark: $i^2 = -1$.

1 Particle plasmon resonance

Plasmonic antennas can scatter light extremely strongly, which renders them interesting for applications such as nano-scale detectors and light trapping in solar cells. However, they also absorb energy in the metal, which is problematic particularly if energy efficiency is of concern. On the other hand, this can also be of use. For example, recently researchers have used the strong absorption in such particles to locally heat and destroy tumor cells [1]. In this exercise we will look at scattering and absorption by dipolar plasmonic nanoparticles.

The static polarizability α of a small sphere ($d \ll \lambda$) with the permitivity ε_1 in the environment ε_2 is given as

$$\alpha = 3V\varepsilon_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \tag{4}$$

where V is the particle volume. For the permittivity of gold, use a Drude model with the following parameters: Plasma frequency $\omega_{p,Au} = 2\pi \cdot 2165$ THz, damping frequency $\gamma_{Au} = 2\pi \cdot 15.9$ THz.

1. Plot the real and imaginary part of $\alpha(\omega)$ of a small gold sphere (diameter: $d = 20$nm; environment $n = 1$).

2. Show that the lineshape of the resonance is Lorentzian for a drude metal, for frequencies ω close to the resonance frequency. What determines the width (Full Width at Half Maximum) of this resonance?

3. What is the extinction / scattering crosssection ($\sigma_{ext}(\omega); \sigma_{scat}(\omega)$) of this particle? Compare it with its geometrical cross-section.

4. Now plot the extinction and scattering crossections of gold particles in air with a diameter of 20, 40 and 60 nm. Compare them to the unitary limit set by the optical theorem. Comment on the result.

5. To make $\alpha(\omega)$ consistent with the optical theorem, one can apply the following recipe:

   $$\frac{1}{\alpha_{\mathrm{dyn}}} = \frac{1}{\alpha_0} - i\frac{k^3}{6\pi\varepsilon_0}, \tag{5}$$

   where α_{dyn} is the new, consistent, *dynamic* polarizability and α_0 is the old, static polarizability. Show that this leads to an added damping term in α_{dyn}.

6. This new damping term is related to radiation. The particle radiates, therefore looses energy, and therefore must experience extra damping.

   The albedo A of a plasmon particle is defined as the ratio of its scattering and its extinction crosssection. Show that A is just the ratio of the radiation damping rate and the total damping rate. Plot the albedo of a gold sphere at plasmon resonance frequency ω_0 as a function of radius.

[1] Rizia Bardhan et al., *Theranostic Nanoshells: From Probe Design to Imaging and Treatment of Cancer*, Accounts of Chemical Research 44 (2011).

7. If one would want to maximize the power absorbed (at resonance) by such a plasmonic particle, e.g. to kill tumor cells, what would be the ideal size?
8. Show, for a metal particle with no intrinsic damping ($\gamma = 0$), that α_{dyn} obeys the unitary limit, i.e.

$$\sigma_{ext}(\omega) \leq \frac{3}{2\pi}\lambda^2 \tag{6}$$

for all frequencies. When is $\sigma_{ext}(\omega)$ equal to the unitary limit?
9. Using α_{dyn}, plot the extinction and scattering crosssections of a gold sphere with 40 nm diameter, and compare to the unitary limit. Do this for a particle with intrinsic damping rate $\gamma = 0\,, 1\,\gamma_{\mathrm{Au}}$ and $2\gamma_{\mathrm{Au}}$. What is the effect of increased damping?

2 The Dimer antenna

Now assume two spherical particles of equal polarizability in vacuum and in close vicinity to each other. Particle i is located at position $\mathbf{r}_i$ and described by its polarizability matrix $\boldsymbol{\alpha}_i$, where $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}$. Both particles are being driven by an electric field $\mathbf{E}_{\mathrm{dr}}(\mathbf{r})$ oriented along the dimer axis, which we choose to be the z-axis. See Fig. 1.

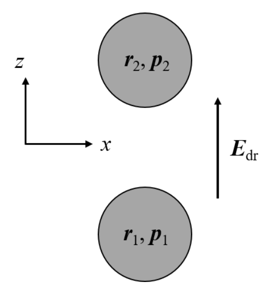

Figure 1: Sketch of the plasmonic dimer antenna.

Because of the symmetry of the problem, we can consider only the zz-component of the particle polarizability matrices and the z-components of the fields. Thus, the dipole moment of particle i is given by the *scalar* equation

$$p_i = \alpha_i E_{\mathrm{tot}}, \tag{7}$$

where $\alpha_i = \hat{z} \cdot \boldsymbol{\alpha}_i \cdot \hat{z}$ is the zz-component of $\boldsymbol{\alpha}_i$. E_{tot} is the z-component of the total field at the particle position, consisting of the driving field $E_{\mathrm{dr}} = \mathbf{E}_{\mathrm{dr}} \cdot \hat{z}$ and the field scattered by the other particle. The latter is given by the scalar version of Eq. (1), i.e.

$$E(\mathbf{r}) = \omega^2 \mu\mu_0 G_{zz}(\mathbf{r}, \mathbf{r}', \omega) \cdot p, \tag{8}$$

where $G_{zz}(\mathbf{r}, \mathbf{r}', \omega)$ is the zz-component of the dyadic Greens function.

1. Set up the 2 coupled linear equations relating the induced dipole moments of the particles to the driving field. Write the equations in matrix form as $\bar{\bar{M}}\mathbf{P} = \mathbf{E}_{\mathrm{dr}}$, with $\mathbf{P}$ the polarization vector $(p_1, p_2)^T$, $\bar{\bar{M}}$ the coupling matrix and $\mathbf{E}_{\mathrm{dr,z}} = (E_{\mathrm{dr}}(\mathbf{r}_1), E_{\mathrm{dr}}(\mathbf{r}_2))^T$ the vector containing the driving field at the positions of both dimer particles. Leave the Green's function unspecified for now.

2. Express the induced dipole moments (p_1, p_2) in terms of the driving field by inverting the coupling matrix $\bar{\bar{M}}$.

3. Diagonalize $\bar{\bar{M}}^{-1}$ to find the eigen-polarizabilities of the dimer. Also find the eigen-modes. What is the total dipole moment for each mode?

4. What is the alignment of the two dipole moments in each respective mode? If I were to drive the dimer with a plane wave propagating in the x-direction, could I excite both modes?

5. Set $\gamma = 0$. At what frequencies do the two eigen-polarizabilities have poles, i.e. where are the hybridized 'resonance frequencies'? Assume that the Green's function is positive and proportional to r^{-3}, with r the separation between particles. How do these pole frequencies depend on separation?

6. If you consider the expression for the scattering cross-section of a particle and think of the dimer as one big particle with an 'effective' total polarizability, which of the two modes do you think will scatter more strongly, and why? What do you think that means for the damping rate (and consequently, linewidth) of the modes? For clarity: do not calculate a result here, just argue.

3 Radiation from an array of dipoles

Antennas are an important tool to transmit electrical signals from A to B by means of electromagnetic waves. The spatial distribution of the (electrical) current density $\mathbf{j}(\mathbf{r}, t)$ depends on both the driving source and the geometry of the antenna and determines the angular radiation pattern. A famous example of an antenna design is the Yagi-Uda antenna. This antenna consists of a finite number of electrical dipoles that are positioned along a line to make the antenna highly directional. In this exercise we will study how an array of dipoles radiates light and how both the position and the drive have an influence on the radiation pattern.

Each electric dipole has size $l \ll \lambda$ and an excitation $p_i \in \mathbb{C}$. We assume the dipoles do not interact. The radiated electric field of a dipole placed at position $\mathbf{r}'$ is given by

$$\psi_t(\mathbf{r}, \omega) = p_i \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cos(\theta_i), \tag{9}$$

where $k = \omega/c = 2\pi/\lambda$ and θ_i is defined in Fig. 2 .

1. Derive the expression for the electric field $\Psi(\mathbf{r}, \omega)$ *within the far-field approximation* for $p_i = p_0$. Which two conditions need to be fulfilled? Numerically check if we can apply the far-field approximation for visible light coming from the sun.

2. Next, we add a second dipole, see Fig. 2. We put dipole 1 at position $\mathbf{r}_1 = (-d/2, 0, 0)$ and dipole 2 at position $\mathbf{r}_2 = (d/2, 0, 0)$, so that their spatial separation is d. Assume dipole 1 and dipole 2 are driven with the same amplitude, but with a phase difference of β. Calculate the radiated field. You will see that the total field equals the field of a single dipole at the origin multiplied by what is often called the *array factor* AF.

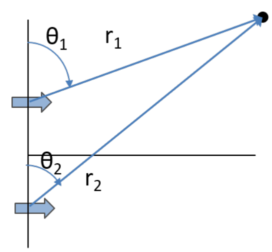

Figure 2: Sketch of two dipoles symmetrically placed around the origin. The black dot is the observation point $\mathbf{r}$.

3. When $\beta = 0$ and $d = \lambda/2$, explain in which directions you expect to have the smallest field amplitude and in which directions the largest? Do the same for the case that $\beta = \pi$ and $d = \lambda/2$.

4. We have N dipoles that are periodically arranged on a line along the x axis with periodicity d, so that the total length of the system $L = (N-1)d$. The dipoles are assumed to be coherently driven, with $p_i = p_0 e^{i\gamma x}$. Calculate the expression for the total electric field. Use the mathematical identity: $\sum_{m=1}^{N} x^m = \frac{x(1-x^N)}{1-x}$.

5. The radiation pattern can be obtained by $r^2|\psi|^2$. For $N = 21$ dipoles periodically placed and driven in phase (so $\gamma = 0$) on a line of length $\lambda/1000$, how do you expect the radiation pattern to look like? How does $|\psi|^2$ scale with the number of dipoles? Give some physical insight in this dependence.

6. Next, we take as the periodicity $d = 0.3\lambda$ and take 7 dipoles, where we assume $\gamma = k$. This situation is very similar to a Yagi-Uda antenna. In which direction does the global maximum of the electric field occur?

Explain why this is true, without any math. How does the directivity qualitatively depend on the total length L of the ensemble of dipoles? Can you explain why the directivity depends on the total L as it does; again no math is needed? Calculate the radiation pattern when $\gamma = 0.5k$? Qualitatively explain the observed change of the radiation pattern.

7. We separate the dipoles a bit more, so that $d = 2.5\lambda$ and $N = 10$ and we set $\gamma = 0$. Before calculating the radiation pattern, how do you expect this emission pattern to look like qualitatively? Determine under which angles we find local maxima of radiation. Which physical phenomenon do we observe here?

4 Single particle polarizability dressed by a lattice

In this exercise we have N identical dipoles periodically placed along the x-axis. A plane wave is incident on the array along the direction $\mathbf{e}_{\text{inc}}$.

1. Give the expression for the electric field at the position of dipole i, where you don't need to include the field produced by the dipole itself. This field is already included in the polarizability $\alpha(\omega)$. You don't need to work out the dyadic Green's function.
2. Write down the equation for the induced dipole moment of dipole i: $\mathbf{p}_i$.
3. Assume the plane wave is perpendicularly incident on the array axis, and its electric field is along the array axis, write down the set of coupled dipoles equations in matrix form: $\bar{\bar{A}}\mathbf{P} = \mathbf{E}$.
4. We next assume to have an infinite array of dipoles. Show that the induced dipole moment of each dipole is given by $p = \frac{\alpha E_0}{1-\alpha S}$ and give the expression for S.
5. The induced dipole moment for the situation of a single dipole would be $p = \alpha E_0$. The expression of the induced dipole moment shows that the polarizability for a dipole in a lattice is "dressed" and given by $\alpha_{\text{dressed}} = \frac{\alpha}{1-\alpha S}$. For a small sphere close to resonance the single-particle polarizability is given by: $\alpha = \frac{-A}{\omega-\omega_p+i\gamma}$, where we take A to be real valued. What is the influence of the lattice on the resonance of the system?