Photonic crystals assignment - 2016

Class dates: *April 26 & 28th 2016*
Due: *May 3rd 2016*
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PART1

In the first part of this assignment, the stop gap width in an infinitely thick periodic medium will be derived. The study case is a quarter wave stack of period Λ: a two-layered material with thicknesses d_1 and d_2 ($\Lambda = d_1 + d_2$) and refractive indices n_1 and n_2 with $n_1, n_2 > 0$, respectively, chosen such that the *optical path length* $\overline{d}$ in both layers is equal.
We define is this document $\overline{k}$ as the wave-vector of light in the quarter-wave stack and ω the frequency (rad/s) of light.

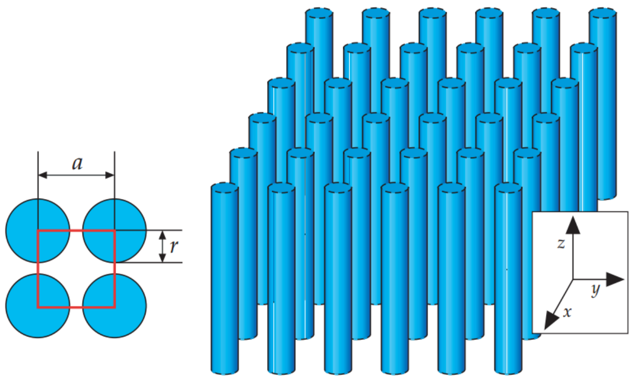
Q1 What is the optical path length $\overline{d}$? Give the dispersion relation for this particular infinite quarter-wave stack (by simplifying the general result obtained in class).

Q2 Give an expression for $\overline{k}$ at the center frequency ω_g of the stop gap defined by $\frac{\omega_g n_{1,2} d_{1,2}}{c} = \frac{\pi}{2}$, assuming that $\overline{k}\,(\omega_g) = \frac{1}{\Lambda}(\pi + \delta)$. Describe qualitatively what happens when a wave is incident *from outside* the slab onto the crystal at that particular frequency.

Q3 Calculate the width $\Delta\omega = \omega_+ - \omega_-$ of the stop gap. To do so, assume $\frac{\omega_\pm n_{1,2} d_{1,2}}{c} = \frac{\pi}{2} + \gamma_\pm$. First explain what value δ takes at the band edges .

PART2

In the second part, we will derive an expression for the photonic bandgap of a square two-dimensional photonic crystal. The photonic crystal in study is pictured in the figure below; the blue cylinders have a different dielectric constant than the regions around it, but all materials are isotropic. This photonic crystal is periodic in the x and y directions and homogeneous in the z-direction. The cylinders have a radius r and pitch a. A field is propagating in the xy-plane.



Q1: Use *symmetry* arguments to explain that the modes sustained by this crystal separate into two distinct polarizations; transverse electric (TE) and transverse magnetic (TM). Give the polarization of **E** and **H** for both these modes .

Hints : the crystal is invariant along the z-direction, the polarization of matter **P** and electric field **E** are related, and a dipole is an anti-symmetric charge distribution by the plane orthogonal to its dipole moment.

From now on we will focus on the electric field component for TM modes, i.e. $\boldsymbol{E} = E_z(x,y)\hat{\boldsymbol{z}}$.

Q2: Starting from Maxwell's equations, in the absence of free charges and currents, derive an eigenvalue problem for the electric field of the TM mode.

In a photonic crystal, the dielectric permittivity is periodic: $\epsilon(\boldsymbol{\rho}) = \epsilon(\boldsymbol{\rho} + \boldsymbol{R})$, where for a square lattice $\boldsymbol{R}$ is any linear combination of the lattice vectors $\boldsymbol{R} = n \cdot a\hat{\boldsymbol{x}} + m \cdot a\hat{\boldsymbol{y}}$,with $\boldsymbol{\rho}$ an arbitrary vector in the x,y-plane and $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ unit vectors and $n, m = 0, \pm 1, \pm 2 \ldots$

Q3: Give reciprocal lattice vectors in the form $\boldsymbol{G} = n \cdot g\hat{\boldsymbol{x}} + m \cdot h\hat{\boldsymbol{y}}$. Give an expression for g and h.

Next is to understand Bloch's theorem. Bloch's theorem states that in a periodic medium, also the eigenfunctions of the electric field have that same periodicity. Then, we can write an ansatz solution for the eigenvalue problem of question 2 as:

$$E_z = \sum_{\boldsymbol{G}} u_{\bar{\boldsymbol{k}},\boldsymbol{G}} e^{i(\bar{\boldsymbol{k}} + \boldsymbol{G})\cdot \boldsymbol{\rho}} \qquad \text{Where } \bar{\boldsymbol{k}} \text{ is the wave vector of the mode.}$$

Q4: Show that this ansatz solution for the electric field has the correct periodicity.

Q5: Derive the following equations for the Fourier coefficients of the electric field:

$$\left(\bar{\boldsymbol{k}} + \boldsymbol{G}\right)^2 u_{\bar{\boldsymbol{k}},\boldsymbol{G}} - \left(\frac{\omega}{c}\right)^2 \sum_{\boldsymbol{G}'} u_{\bar{\boldsymbol{k}},\boldsymbol{G}-\boldsymbol{G}'} \tilde{\epsilon}_{\boldsymbol{G}'} = 0$$

Where $\epsilon_{\boldsymbol{G}'}$ are Fourier coefficients of the relative permittivity function, i.e.:

$$\epsilon(\boldsymbol{\rho}) = \sum_{\boldsymbol{G}} \tilde{\epsilon}_{\boldsymbol{G}} e^{i\boldsymbol{G}\cdot\boldsymbol{\rho}} \quad \text{with} \quad \tilde{\epsilon}_{\boldsymbol{G}} = \frac{1}{V}\int_V \epsilon(\boldsymbol{\rho}) e^{-i\boldsymbol{G}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho}$$

Hints : To avoid renaming errors for variables' indices, you should properly write down the Fourier transform you perform to obtain the equation for a Fourier coefficient.

To describe wave propagation in the crystal, it is necessary to find a dispersion relation; a relation between ω and $\tilde{\boldsymbol{k}}$. Solving equation from question 5 can both yield the dispersion and the eigenfunctions $u_{\bar{\boldsymbol{k}},\boldsymbol{G}}$. This equation is in principle an infinite set of coupled equations, but in practice only a few equations are needed to provide an approximate result.

Q6: Now, assume propagation along $\hat{\boldsymbol{x}}$ ($\tilde{\boldsymbol{k}} = \tilde{k}\hat{\boldsymbol{x}}$ and we consider only $\boldsymbol{G} = n \cdot g\hat{\boldsymbol{x}}$). Expand the permittivity function only up to the first three Fourier coefficients: $\tilde{\epsilon}_0$, $\tilde{\epsilon}_g$ and $\tilde{\epsilon}_{-g}$. Set up *a two-band model* and use that near the Bragg condition, only the terms $u_{\bar{k},0}$ and $u_{\bar{k},-g}$ are dominant, and derive the following dispersion relation:

$$\left(\frac{\omega}{c}\right)^4 \left(\tilde{\epsilon}_0^2 - \tilde{\epsilon}_g\tilde{\epsilon}_{-g}\right) - \left(\frac{\omega}{c}\right)^2 \tilde{\epsilon}_0 \left(\left(\bar{k} - g\right)^2 + \bar{k}^2\right) + \bar{k}^2\left(\bar{k} - g\right)^2 = 0$$

Q7: Show that when the Bragg condition is exactly satisfied, two roots for ω^2 may be obtained:

$$\omega_+^2 = \frac{\pi^2 c^2}{a^2(\tilde{\epsilon}_0 - |\tilde{\epsilon}_g|)} \quad \text{and} \quad \omega_-^2 = \frac{\pi^2 c^2}{a^2(\tilde{\epsilon}_0 + |\tilde{\epsilon}_g|)}$$

The frequencies between ω_+ and ω_- are within the bandgap, that has arisen from the coupling between the wave components $u_{\bar{k},0}$ and $u_{\bar{k},-g}$.

Q8: The fractional bandgap $f_g = \frac{\omega_+ - \omega_-}{\omega_g}$ is a good measure of the 'photonic strength' of the crystal, i.e., the coupling strength between $u_{\bar{k},0}$ and $u_{\bar{k},-g}$ at the Bragg condition.
Plot on the same graph the fractional bandgap f_g against the filling ratio $^r/_a$ for the 2D square lattice consisting of *pillars of silicon (*$\epsilon = 12$*) in air,* and *pillars of air in silicon* . Comment on what makes a strong photonic bandgap.

Hints: To avoid Bessel functions you may approximate the cylinders as squares. You can use that the center frequency in the stop gap is at $\omega_g = \pi c/(a\sqrt{\tilde{\epsilon}_0})$.